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LETTERS TO THE EDITOR

NATURAL FREQUENCIES OF AXIALLY TRAVELLING TENSIONED BEAMS IN CONTACT WITH A STATIONARY MASS

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1. INTRODUCTION

The problem of axially travelling materials is of technological interest since many such materials are observed in many places. Some examples are rolled steel belts, high-speed magnetic tapes, band-saws, power transmission chains and belts, aerial cable tramways, flexible robotic manipulators with prismatic joints, appendages under deploying motion, pipes and beams conveying fluid, etc. A vast literature can be found cited in references [1, 2].

Here are some studies about axially travelling second order continua with mass. Wickert and Mote [3] modelled a monocable ropeway system as an axially travelling string that transports an attached mass particle and investigated frequency and amplitude variation by using the method of strained parameters. The authors [4] extended the classical travelling load problem to string systems in which the guideway has also prescribed axial motion. Yang [5] presented eigenvalue inclusion principle for the translating string under non-dissipative, pointwise constraints. Zhu and Mote [6] determined the transient response of constrained translating strings under arbitrary disturbances. Chen [7] studied a travelling string problem in contact with a stationary load system and investigated the effects of constraints on the natural frequencies and stability of the system. Zhu *et al.* [8] developed a new spectral analysis for the asymptotic locations of eigenvalues of a constrained translating string and determined the asymptotic solutions from the characteristic equation of the coupled system of constraint and string for all constraint parameters. Lee and Mote [9] analyzed energy transfer and mode localization in a translating string coupled to a stationary system using travelling waves. Öz and Evrensel [10] and Öz [11] calculated natural frequencies and amplitude variations of highly tensioned pipes conveying fluid with a stationary mass.

There are some studies about fourth order axially moving systems with mass constraints. Stylianou and Tabarrok [12, 13] used finite element formulation to consider translational and rotary inertia effects of the tip mass and made a stability analysis. Borglund [14] considered the stability and optimal design of a beam subject to forces induced by fluid flow through attached pipes with a tip mass by using finite element formulation. Fung *et al.* [15] obtained the equations of motion for a deploying beam with a tip mass by using Hamilton's principle. Kang [16] investigated the effect of rotary inertia of a concentrated mass on the natural frequencies of a fourth order pipe by using Galerkin method. Zhu *et al.* [17] made a spectral analysis determining asymptotically the

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distribution of eigenvalues of a constrained, translating, tensioned beam in closed form. Kang [18] studied effects of rotary inertia of concentrated masses on the natural vibrations of a fluid-conveying pipe using Galerkin's method.

In this study, the transverse vibration of highly tensioned Euler–Bernoulli beams is investigated. The beam is in contact with a stationary mass. The axial velocity is assumed

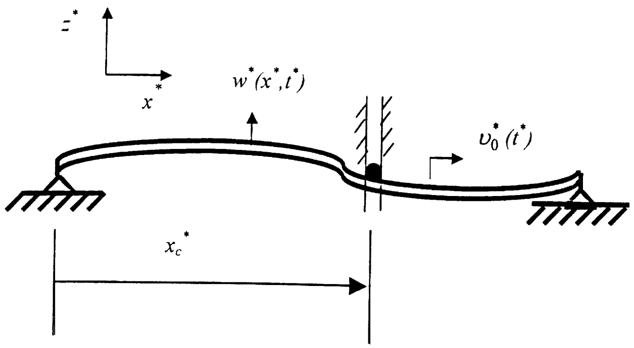

Figure 1. Schematics of the axially travelling tensioned beam on simple supports with a stationary mass.

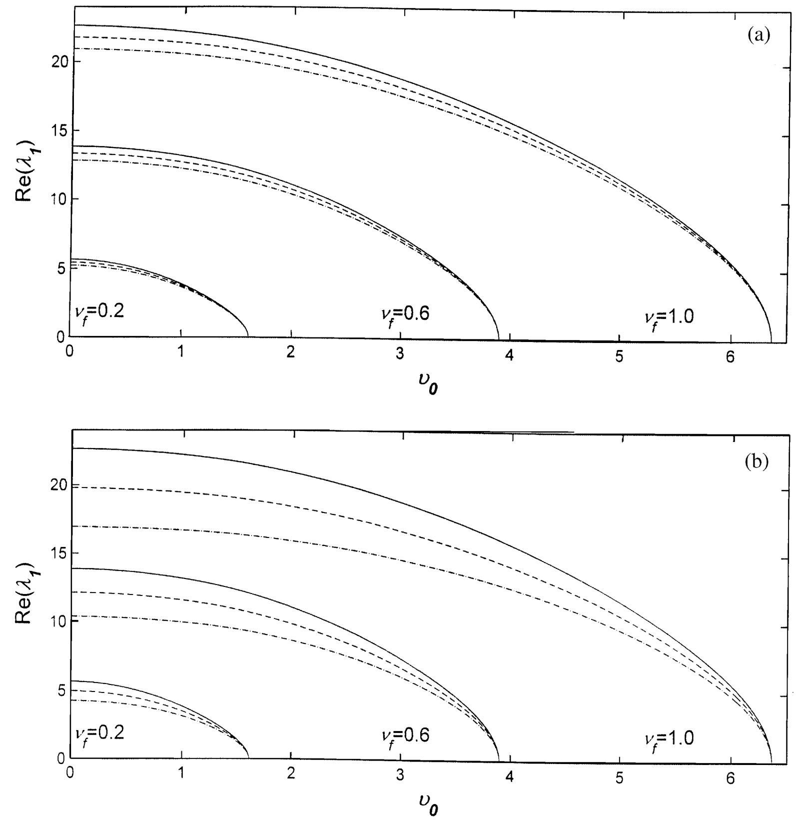

Figure 2. The axial velocity versus natural frequency of the axially travelling tensioned beam on fixed supports ($n = 1$, $\alpha = 0$ (——), 0·1 (– – –), 0·2 (–·–·–)); (a) $x_c = 0{\cdot}25$, (b) $x_c = 0{\cdot}50$.

to be constant. The linear equations of motion are solved analytically by means of direct application of the strained parameters method (a perturbation technique) for fixed and simply supported cases. The natural frequencies are calculated depending on flexural stiffness parameters and mean velocities. Assuming the stationary mass as small, the correction term is calculated at the second order of perturbation analysis and perturbed natural frequencies are obtained. The effect of mass on different locations is investigated. The mass reduces the frequencies, but the amplitudes of the vibration stay constant.

2. EQUATIONS OF MOTION

For the axially travelling tensioned beam shown in Figure 1, x^* and z^* are the spatial co-ordinates, t^* is the time, w^* is the transverse displacement, υ_0^* is the constant axial velocity, m_b is the mass per unit length, and EI is the flexural rigidity. Tension force applied to the beam is P_0. The length of beam between supports is L. Assuming an Euler–Bernoulli beam with small transverse displacement compared with length L and sufficiently large tension force compared with the effects arising from elongation, and denoting the derivatives with respect to the spatial variable by $(\,)'$ and the derivatives with

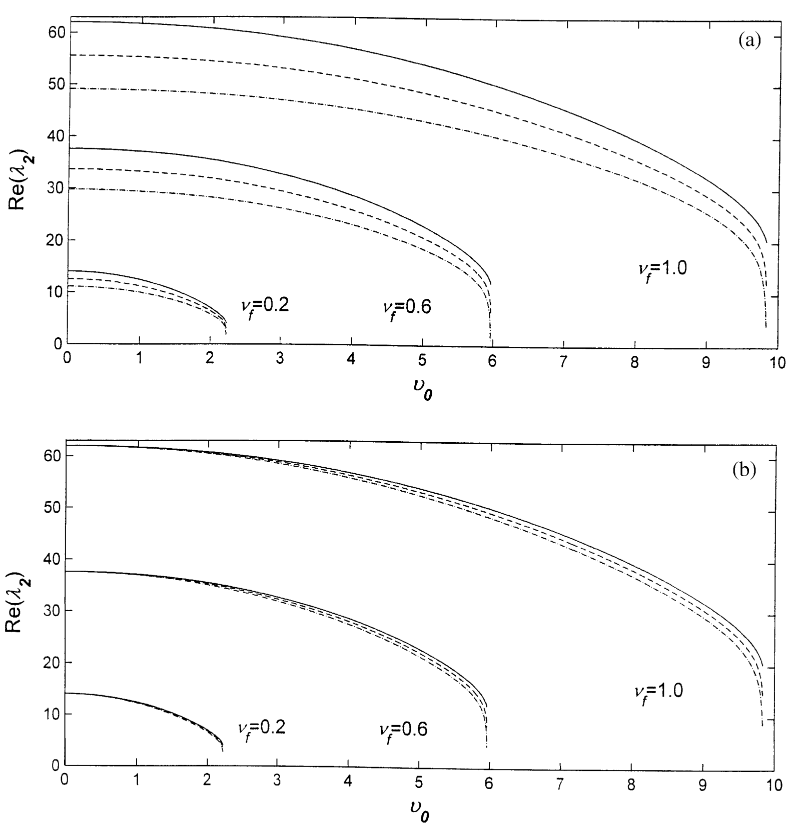

Figure 3. The axial velocity versus natural frequency of the axially travelling tensioned beam on fixed supports ($n = 2$, $\alpha = 0$ (——), 0·1 (– – –), 0·2 (–·–·–)); (a) $x_c = 0{\cdot}25$, (b) $x_c = 0{\cdot}50$.

respect to time by $(\cdot)$, the linear equation of transverse motion of the travelling tensioned beam can be written as [19]

$$m_b(\ddot{w}^* + 2v_0^*\dot{w}^{*'} + v_0^{*2}w^{*''}) - P_0 w^{*''} + EIw^{*\mathrm{iv}} = 0. \tag{1}$$

According to Pan [20] and Sato *et al.* [21], the effect of stationary mass placed at $x^* = x_c^*$ can be modelled as

$$m_c\delta(x^* - x_c^*)\ddot{w}^*, \tag{2}$$

where m_c is the stationary mass and δ is the Dirac delta function. This equation represents the inertia force due to the lateral acceleration of the stationary mass. Then, the governing equation for the tensioned beam travelling with constant velocity in contact with a stationary mass can be expressed as

$$\{m_b + m_c\delta(x^* - x_c^*)\}\ddot{w}^* + 2m_b v_0^*\dot{w}^{*'} + m_b v_0^{*2}w^{*''} - P_0 w^{*''} + EIw^{*\mathrm{iv}} = 0 \tag{3}$$

and the fixed–fixed and simple–simple boundary condition are

$$w^*(0, t^*) = w^*(L, t^*) = 0, w^{*'}(0, t^*) = w^{*'}(L, t^*) = 0, \tag{4}$$

$$w^*(0, t^*) = w^*(L, t^*) = 0, w^{*''}(0, t^*) = w^{*''}(L, t^*) = 0 \tag{5}$$

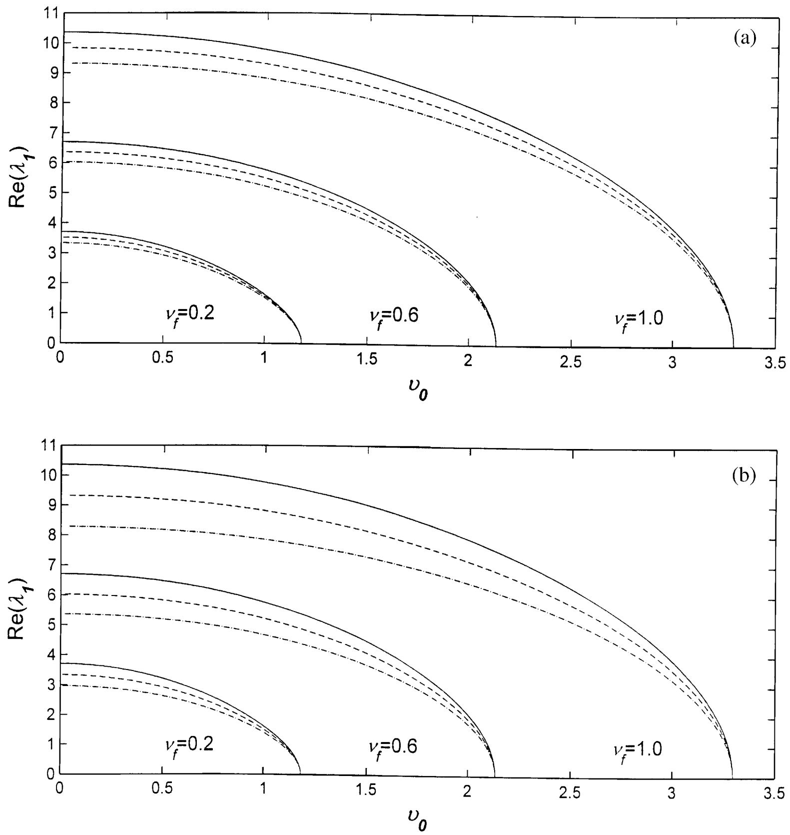

Figure 4. The axial velocity versus natural frequency of the axially travelling tensioned beam on simple supports ($n = 1$, $\alpha = 0$ (——), 0·1 (– – –), 0·2 (– · – · –)); (a) $x_c = 0{\cdot}25$, (b) $x_c = 0{\cdot}50$.

respectively. Introducing dimensionless parameters, the equation of motion and boundary conditions become

$$\{1 + \alpha\delta(x - x_c)\}\ddot{w} + 2\upsilon_0\dot{w}' + (\upsilon_0^2 - 1)w'' + \upsilon_f^2 w^{\text{iv}} = 0, \tag{6}$$

$$w(0, t) = w(1, t) = 0, \qquad w'(0, t) = w'(1, t) = 0, \tag{7}$$

$$w(0, t) = w(1, t) = 0, \qquad w''(0, t) = w''(1, t) = 0 \tag{8}$$

and the dimensionless parameters are

$$\alpha = \frac{m_c}{Lm_b} \quad w = \frac{w^*}{L}, \quad x = \frac{x^*}{L}, \quad x_c = \frac{x_c^*}{L}, \quad t = \frac{t^*}{L}\sqrt{\frac{P_0}{m_p}}, \quad \upsilon_0 = \upsilon_0^*\sqrt{\frac{m_b}{P_0}}, \quad \upsilon_f = \sqrt{\frac{EI}{P_0 L^2}}. \tag{9}$$

In equation (6) $\ddot{w}$, $2\dot{w}'\upsilon_0$ and $\upsilon_0^2 w''$ denote local, Coriolis and centrifugal acceleration components respectively. υ_f is the flexural stiffness parameter.

3. PERTURBATION ANALYSIS

Solutions of the approximate eigenvalue problem are restricted to systems in which the mass ratio, α, is small. The transition of solutions from those of the axially travelling

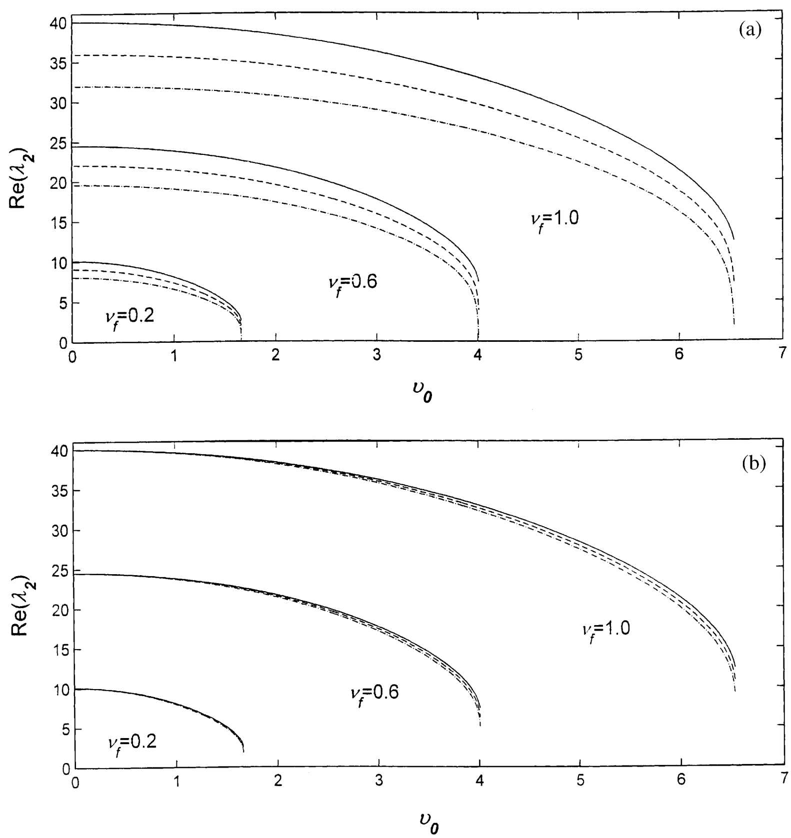

Figure 5. The axial velocity versus natural frequency of the axially travelling tensioned beam on simple supports ($n = 2$, $\alpha = 0$ (——), 0·1 (– – –), 0·2 (– · – · –)); (a) $x_c = 0{\cdot}25$, (b) $x_c = 0{\cdot}50$.

tensioned beam to those of the axially travelling tensioned beam in contact with a stationary mass system is studied by the method of strained parameters to determine a first order perturbation solution for small α. Let us assume the solution to equation (6) as

$$w(x,t) = Y_n(x)e^{i\lambda_n t} + cc, \tag{10}$$

where $i = \sqrt{-1}$, cc is the complex conjugate. We can make the expansions for both the complex eigenfunction Y_n and eigenvalue λ_n,

$$Y_n(x,\alpha) = Y_n^{(0)}(x) + \alpha Y_n^{(1)}(x) + O(\alpha^2), \tag{11}$$

$$\lambda_n = \lambda_n^{(0)} + \alpha\lambda_n^{(1)} + O(\alpha^2). \tag{12}$$

Substituting equations (10)–(12) into equation (6) and using boundary conditions (7) and (8), one obtains coupled equations in two orders of α by neglecting higher order of perturbations.

Order α^0:

$$v_f^2 Y_{\mathrm{n}}^{(0)\mathrm{iv}} + (v_0^2 - 1)Y_n^{(0)''} + 2\mathrm{i}v_0\lambda_n^{(0)}Y_0^{(0)'} - \lambda_n^{(0)2}Y_n^{(0)} = 0, \tag{13}$$

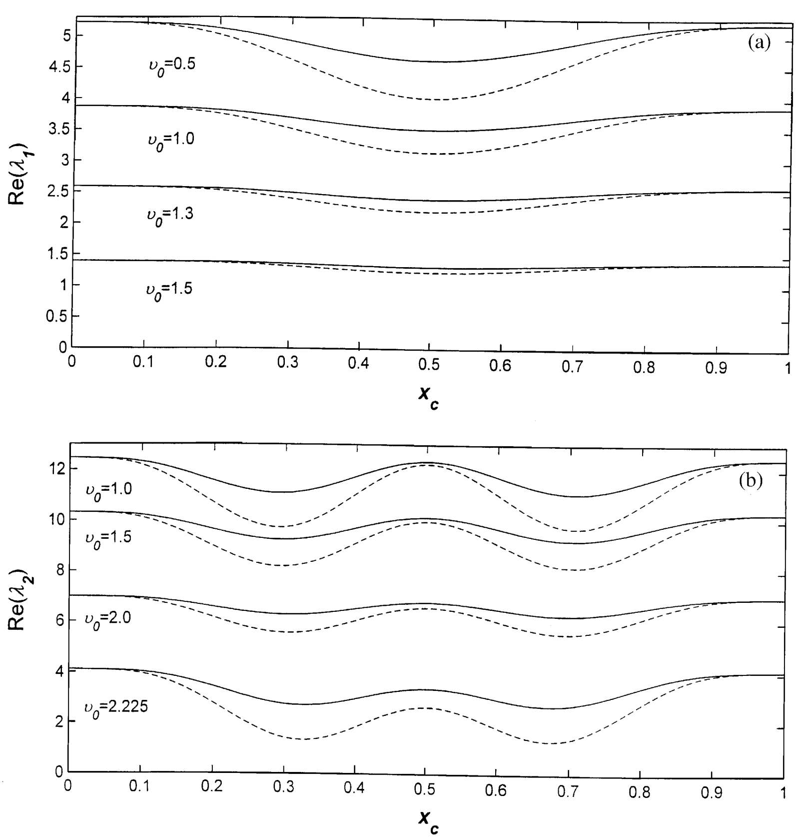

Figure 6. Natural frequency of the axially travelling tensioned beam on fixed supports versus the position of the stationary mass ($v_f = 0{\cdot}2$, $\alpha = 0{\cdot}1$, (——), 0·2 (– – –)).

$$Y_n^{(0)}(0) = 0, \quad Y_n^{(0)}(1) = 0, \quad Y_0^{(0)'}(0) = 0, \quad Y_n^{(0)'}(1) = 0, \tag{14}$$

$$Y_n^{(0)}(0) = 0, \quad Y_n^{(0)}(1) = 0, \quad Y_n^{(0)''}(0) = 0, \quad Y_n^{(0)''}(1) = 0. \tag{15}$$

Order α^1:

$$v_f^2 Y_n^{(1)\mathrm{iv}} + (v_0^2 - 1) Y_n^{(1)''} + 2\mathrm{i}v_0 \lambda_n^{(0)} Y_n^{(1)'} - \lambda_n^{(0)2} Y_n^{(1)} = 2\lambda_n^{(1)} \left(\lambda_n^{(0)} Y_n^{(0)} - \mathrm{i}v_0 Y_n^{(0)'}\right)$$

$$+ \delta(x - x_c)\lambda_n^{(0)2} Y_n^{(0)}, \tag{16}$$

$$Y_n^{(1)}(0) = 0, \quad Y_n^{(1)}(1) = 0, \quad Y_n^{(1)'}(0) = 0, \quad Y_n^{(1)'}(1) = 0. \tag{17}$$

$$Y_n^{(1)}(0) = 0, \quad Y_n^{(1)}(1) = 0, \quad Y_n^{(1)''}(0) = 0, \quad Y_n^{(1)''}(1) = 0. \tag{18}$$

The solution of equation (13) can be assumed as

$$Y_n^{(0)}(x) = c_{1n}(\mathrm{e}^{k_{1n}x} + C_{2n}\mathrm{e}^{k_{2n}x} + C_{3n}\mathrm{e}^{k_{3n}x} + C_{4n}\mathrm{e}^{k_{4n}x}), \tag{19}$$

where k_{sn} is the wave number in the nth mode. Substituting the shape function into equation of order α^0, the dispersion relation is obtained as

$$v_f^2 k_{sn}^4 + (1 - v_0^2)k_{sn}^2 - 2v_0\lambda_n^{(0)} k_{sn} - \lambda_n^{(0)2} = 0, \quad s = 1, 2, 3, 4, \; n = 1, 2, \ldots \tag{20}$$

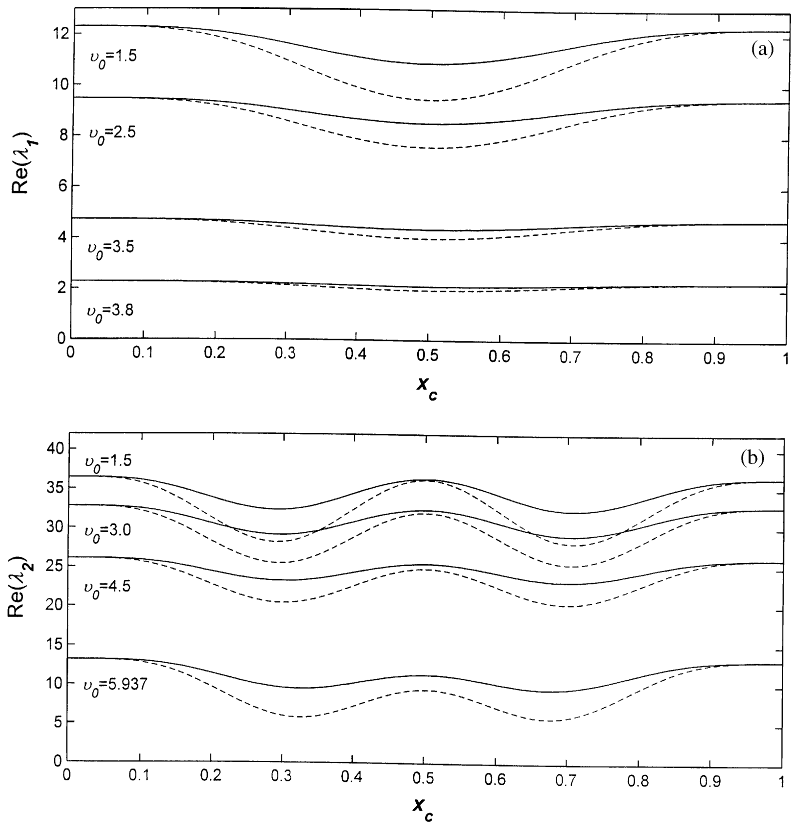

Figure 7. Natural frequency of the axially travelling tensioned beam on fixed supports versus the position of the stationary mass ($v_f = 0{\cdot}6$, $\alpha = 0{\cdot}1$, (——), 0·2 (- - - - -)).

and from the solution of boundary conditions one obtains the support conditions for fixed–fixed and simple–simple case [19, 22, 23],

$$[e^{i(k_{1n}+k_{2n})} + e^{i(k_{3n}+k_{4n})}](\Gamma_{1n} - \Gamma_{2n})(\Gamma_{3n} - \Gamma_{4n})$$
$$+ [e^{i(k_{1n}+k_{3n})} + e^{i(k_{2n}+k_{4n})}](\Gamma_{2n} - \Gamma_{4n})(\Gamma_{3n} - \Gamma_{1n})$$
$$+ [e^{i(k_{2n}+k_{3n})} + e^{i(k_{1n}+k_{4n})}](\Gamma_{1n} - \Gamma_{4n})(\Gamma_{2n} - \Gamma_{3n}) = 0, \tag{21}$$

where $\Gamma_s = k_s$ for fixed–fixed and $\Gamma_s = k_s^2$ for simple–simple end conditions. The simultaneous solution of equations (20) and (21) will give velocity-dependent natural frequencies without stationary mass for different flexural stiffness coefficients [22, 23]. In section 4 some numerical examples will be given. From the boundary conditions, the shape function is obtained as

$$Y_n(x) = c_{1n}\Bigg(e^{ik_{1n}x} - \frac{(\Gamma_{4n} - \Gamma_{1n})(e^{ik_{3n}} - e^{ik_{1n}})}{(\Gamma_{4n} - \Gamma_{2n})(e^{ik_{3n}} - e^{ik_{2n}})}e^{ik_{2n}x} - \frac{(\Gamma_{4n} - \Gamma_{1n})(e^{ik_{2n}} - e^{ik_{1n}})}{(\Gamma_{4n} - \Gamma_{3n})(e^{ik_{2n}} - e^{ik_{3n}})}e^{ik_{3n}x}$$
$$+ \left(-1 + \frac{(\Gamma_{4n} - \Gamma_{1n})(e^{ik_{3n}} - e^{ik_{1n}})}{(\Gamma_{4n} - \Gamma_{2n})(e^{ik_{3n}} - e^{ik_{2n}})} + \frac{(\Gamma_{4n} - \Gamma_{1n})(e^{ik_{2n}} - e^{ik_{1n}})}{(\Gamma_{4n} - \Gamma_{3n})(e^{ik_{2n}} - e^{ik_{3n}})}\right)e^{ik_{4n}x}\Bigg), \tag{22}$$

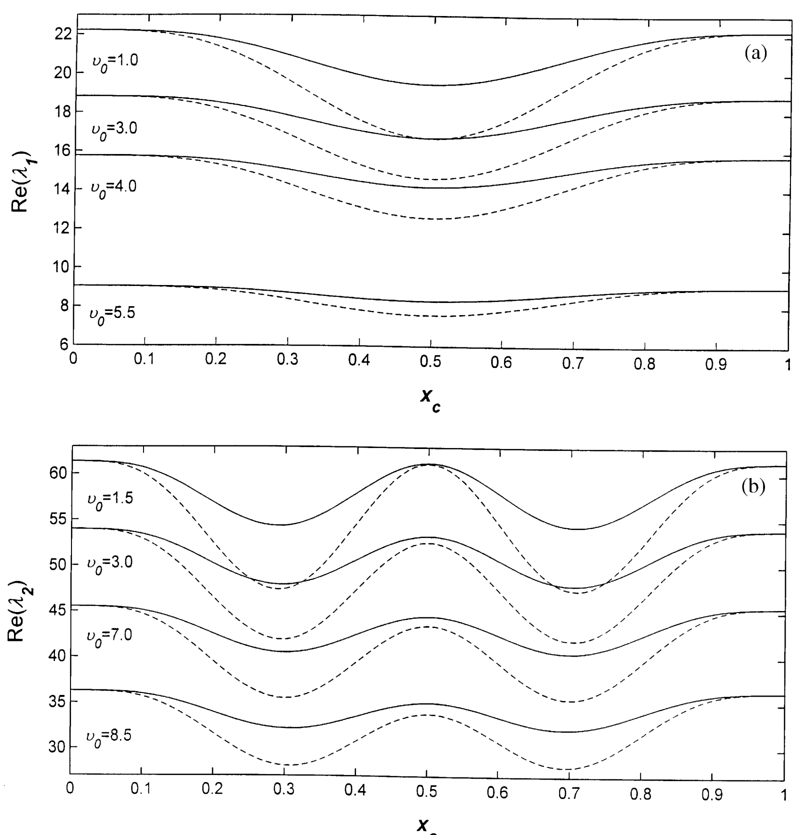

Figure 8. Natural frequency of the axially travelling tensioned beam on fixed supports versus the position of the stationary mass ($v_f = 1{\cdot}0, \alpha = 0{\cdot}1$, (——), 0·2 (-----)).

where $\Gamma_s = k_s$ for fixed–fixed and $\Gamma_s = k_s^2$ for simple–simple end conditions. The solution of order α^1 will give a correction term to natural frequency of axially travelling tensioned beam in contact with a stationary mass. Since the homogeneous problem (13) has a non-trivial solutions, the inhomogeneous equation (16) has a non-secular solution if and only if the following solvability condition is satisfied (see reference [24]).

$$\int_0^1 \{2\lambda_n^{(1)}[\lambda_n^{(0)} Y_n^{(0)} - \mathrm{i}\upsilon_0 Y_n^{(0)'}] + \delta(x - x_c)\lambda_n^{(0)2} Y_n^{(0)}\} \bar{Y}_n^{(0)} \,\mathrm{d}x = 0, \tag{23}$$

where $\bar{Y}_n^{(0)}$ is the complex conjugate of the shape function (22). The correction term $\lambda_n^{(1)}$can be obtained from the solvability condition. The perturbed eigenvalue of the axially travelling tensioned beam in contact with a stationary mass constraint from equations (12) and (23) is

$$\lambda_n = \lambda_n^{(0)} + \alpha \frac{\lambda_n^{(0)2} | Y_n^{(0)} \bar{Y}_n^{(0)} |_{x=x_c}}{2\left\{\mathrm{i}\upsilon_0 \int_0^1 Y_n^{(0)} \bar{Y}_n^{(0)} \,\mathrm{d}x - \lambda_n^{(0)} \int_0^1 Y_n^{(0)} \bar{Y}_n^{(0)} \,\mathrm{d}x\right\}}. \tag{24}$$

The real part, $\mathrm{Re}(\lambda_n)$, of the perturbed eigenvalue corresponds to the frequency of oscillation and the imaginary part is related to amplitude variation.

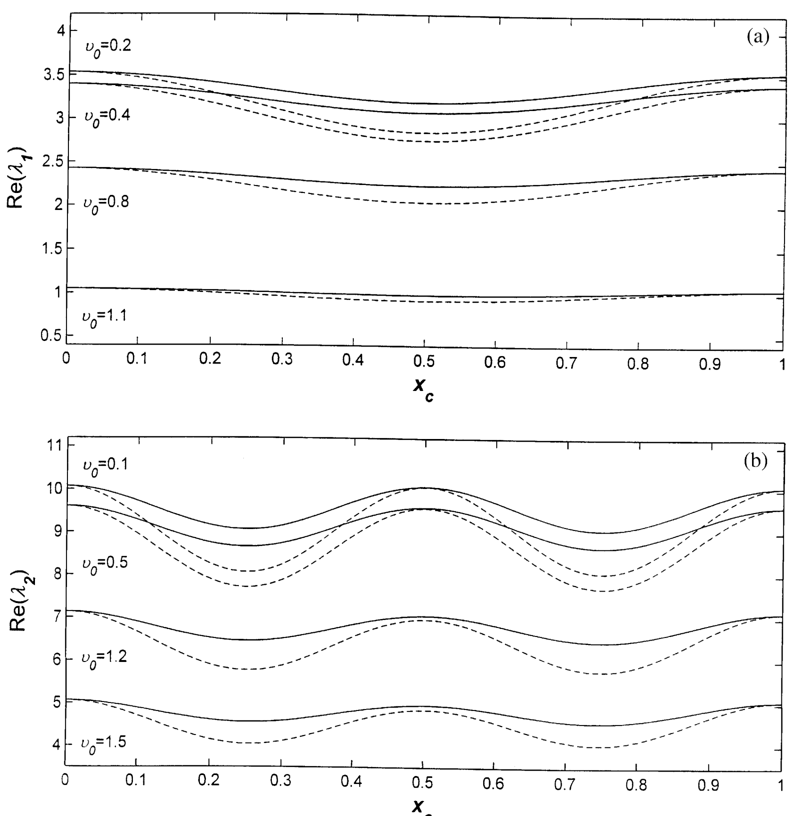

Figure 9. Natural frequency difference due to stationary mass of the axially travelling tensioned beam on simple supports versus the position of the stationary mass ($\upsilon_f = 0{\cdot}2, \alpha = 0{\cdot}1$, (——), 0·2 (– – –)).

4. NUMERICAL ANALYSIS

By solving dispersion equation (20) and support condition (21) simultaneously, the velocity-dependent natural frequencies can be obtained. The variation of frequency or correction term due to stationary mass ratio α, can be obtained from the real part of the second term in equation (24). The natural frequency versus velocity is presented in Figures 2 and 3 for fixed supports and in Figures 4 and 5 for simple supports. The frequency values are plotted for three different flexural stiffness parameters (0·2, 0·6, 1·0) and two stationary mass ratios (0·1, 0·2), when the mass is located at $x_c = 0{\cdot}25$ or 0·50. The stationary mass does not change the critical velocity values, but decreases the frequencies in all velocities. The greater the mass, the lower the frequency. In the first mode, the effect of the mass is larger when $x_c = 0{\cdot}50$ where larger displacement takes place as shown in Figures 2 and 4 for fixed and simply supported moving beams respectively. In Figures 3 and 5, the second mode is presented for fixed and simply supported cases. As seen from the figures, when the mass is located at $x_c = 0{\cdot}25$, its effect is larger at low velocities and it decreases as the velocity is increased. Near critical velocity values, again the frequencies are greatly affected. When the mass is located at $x_c = 0{\cdot}50$, the mass has less effect in small velocities, but its effect increases in higher velocities.

In Figures 6–8, the variation of frequency with the position of mass constraint are plotted for different travelling velocities υ_0, mass ratios, α and flexural stiffness parameters

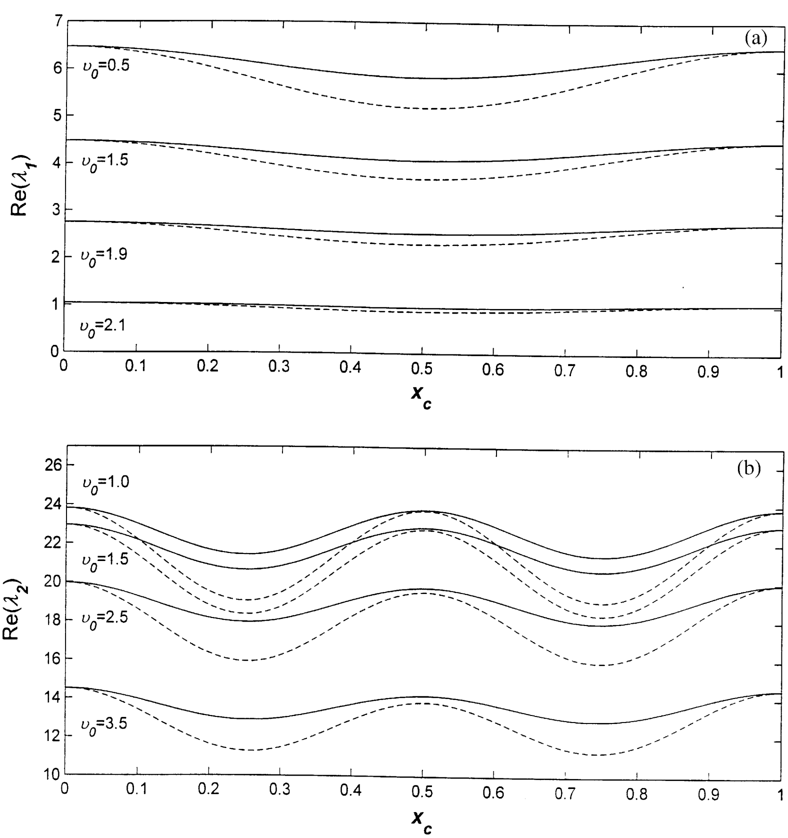

Figure 10. Natural frequency of the axially travelling tensioned beam on simple supports versus the position of the stationary mass ($v_f = 0{\cdot}6, \alpha = 0{\cdot}1$, (——), 0·2 (– – –)).

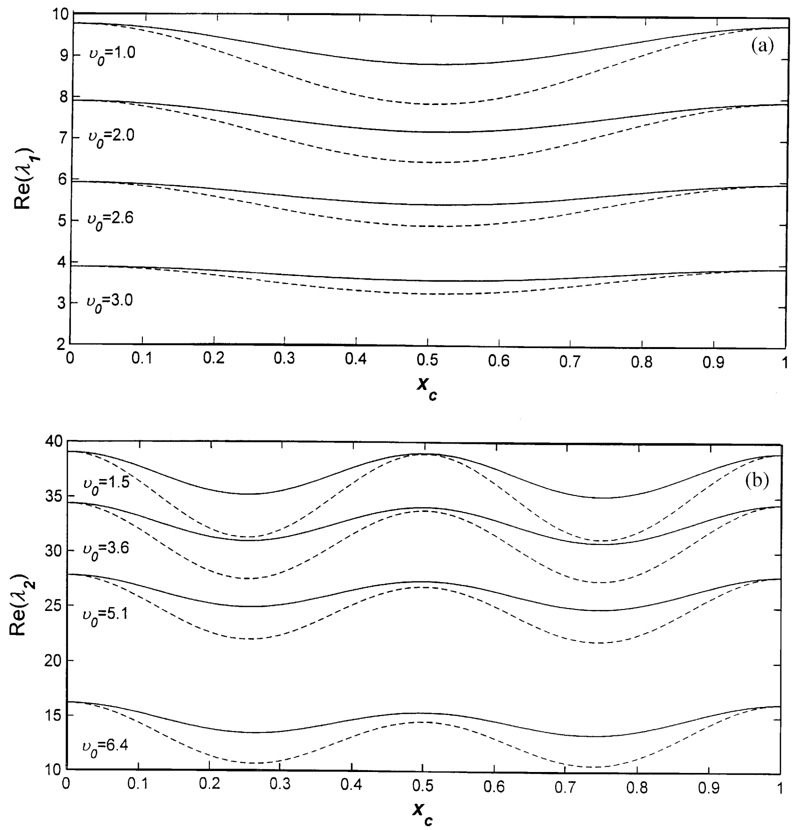

Figure 11. Natural frequency of the axially travelling tensioned beam on simple supports versus the position of the stationary mass ($v_f = 1{\cdot}0, \alpha = 0{\cdot}1$, (——), 0·2 (– – –)).

$v_f(0{\cdot}2, 0{\cdot}6, 1{\cdot}0)$, respectively, for the first and second modes for the fixed–fixed case. In the first mode, Re(λ_1), when the mass is located in the middle, its effect is larger especially at low velocities. In the second mode, Re(λ_2), inclusion of the mass near middle point has less effect on the frequency, but the effect increases at high velocities. The largest variation on the frequencies takes place around $x_c = 0{\cdot}30$ and 0·70 for this mode. Similar conclusions can be drawn for simply supported case from Figures 9–11. The imaginary parts of all eigenvalues, Im(λ_n), which represent amplitude variation due to a stationary mass, are not affected and always remain zero. In other words, the stationary mass does not affect the stability of the system for the given two end conditions.

5. CONCLUSIONS

The linear transverse vibration of highly tensioned Euler–Bernoulli beams travelling axially in contact with a stationary mass constraint is considered. The beam has a constant axial velocity. The equations of motion are solved analytically applying the strained parameters method for fixed–fixed and simple–simple end conditions. The solutions depend on the axial position of the mass constraint. The natural frequencies are calculated depending on axial velocity and stationary mass for different flexural stiffness parameters.

The stationary mass constraint decreases the frequencies through all travelling velocities. It does not change the critical velocity of the travelling beam. The variation of frequency with position of mass constraint is plotted for different parameters. No amplitude variation is found due to stationary mass for fixed and simply supported cases.

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